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Transient thermal conduction analysis of a rectangular plate with multiple insulated cracks by the alternating method

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Abstract

An efficient analytical alternating method is developed in this study to investigate the transient thermal conduction problem of a finite plate with multiple insulated cracks. Analytical solutions of rectangular plates subjected to a point temperature and a point temperature gradient on boundaries are derived to construct the full field solutions of rectangular plates under arbitrarily distributed thermal loadings by Gauss integration. By using these analytical fundamental solutions, the analytical alternating method is applied to obtain the full field temperature distribution of rectangular plate with multiple cracks. The temperature distribution of a finite plate with multiple insulated cracks in steady state are compared with the results obtained by other researches and excellent agreements are shown. The transient temperature distribution of a finite plate with multiple insulated cracks are obtained and discussed in detail. © 2001 Elsevier Science Ltd. All rights reserved.

1. Introduction

The safety assessment of structures in harsh thermal environments is of increasing design engineers. It is known that components work under high temperature variation usually give rise to defects or cracks. When the structure is subjected to an improper thermal condition, the heat flow will be disturbed by the cracks. The high intensification of the transient temperature gradient will induce thermal stress that may cause rapid linkage of several small cracks into one large crack. Although each individual crack may be considered safe within the damage tolerance of the structure, damages to the structure can be caused by the interaction of multiple cracks in the structure. Hence the development of analyzing methods that can accurately estimate the temperature distribution of structures with multiple cracks is needed.

In the literature, there are many available results for the transient analysis of a cracked plate under thermal boundary conditions. Emery et al. [1] computed the transient thermal stress intensity factor (TSIF) for an edge crack in a finite plate subjected to heat flow by FEM. Nied [2] discussed the problem of an edge-cracked strip with convective cooling or heating on the side of the plate containing the crack and insulated on the other side. It was shown that surface heating might induce compressive transient thermal stress in the plate surface, which will force the crack surface contact together over a certain length. Similar problem was also studied by Rizk [3,4] and unique results were obtained. A ramp function that is more realistic than a step function was assumed at the boundary by Rizk [4]. Rizk and Radwan [5] studied a cracked semi-infinite plate subjected to a sudden cooling on the surface in the form of a ramp function. Kokini and Reynolds [6] analyzed the transient behavior of an interface crack located at the center or edge of two finite dissimilar materials under thermal boundary conditions by FEM. Lee and Hong [7] computed the transient TSIF for a finite plate with a central cusp crack by BEM. Magalhaes and Emery [8] studied the transient effect of thermal boundary conditions on the propagation of cracks in a brittle substrate caused by residual tension in a brittle film using a finite element approach. As the cases mentioned above, the subject

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Nomenclature

a	half length of the crack [m]
f, g	temperature gradient [°C/m]
F	initial temperature in the plate [°C]
h	half height of the rectangular plate [m]
l	half length of the square plate [m]
q	heat flux vector [W/m^2]
t	time [s]
T	temperature [°C]
w	half width of the rectangular plate [m]
x, y	Cartesian coordinates [m]
X, Y	eigenfunctions
z	complex variable
$Z_{\mathrm{III}}, \bar{Z}_{\mathrm{III}}$	complex functions

Greek symbols

α	thermal diffusivity [m^2/s]
β, γ	eigenvalues
δ	unit impulse function
κ	thermal conductivity [W/(m°C)]
θ	the inclined angle of the crack

Subscripts

b	bottom boundary of the plate
ci	the crack i
h	homogeneous transient problem
l	left boundary of the plate
r	right boundary of the plate
s	steady state problem
t	top boundary of the plate

seems limited to the transient thermal fracture analysis of a finite structure with a single crack. The transient analysis of a finite body with multiple cracks under thermal boundary conditions has not been discussed yet due to the great complexity of the problem.

Tsai and Ma [9] developed a new formulation of thermal weight function to determine TSIF for the thermal fracture problem. The thermal weight function is a universal function for a given cracked geometry and is independent of applied loads. For a finite cracked structure, if the thermal weight function is obtained from a simple loading case, the TSIF can be computed by integrating the thermal weight function and temperature distribution. This method can be extended to evaluate the transient TSIF by replacing the temperature distribution with transient temperature distribution.

The purpose of this work is to develop an accurate and efficient method to analyze the transient thermal conduction problem of a finite plate with arbitrarily located multiple insulated cracks. For the steady-state case, Chen and Chang [10] employed a finite element alternating method to calculate the temperature distribution on a finite plate with multiple insulated cracks. The same problem was also studied by the boundary element alternating method by Chen and Tu [11].

In the analysis of thermoelastic fracture problem, the analytical solutions are available only for simple problems with specific boundary conditions. The finite element method is a powerful numerical technique for thermal fracture analysis, but the accuracy of the solutions computed by FEM is largely mesh-dependent. The Schwartz–Neumann alternating technique [12] was introduced to overcome the shortcomings of FEM in dealing with multiple crack problems. Essentially, the alternating method is a linear super position method. The complicated cracked finite body solution can be obtained by iterating between the solution for the non-crack finite body, and the solution for an infinite body with a crack (or cracks). Various methods are used to solve the sub-problems of non-crack body lead to various characteristics of Schwartz–Neumann alternating methods.

The analytical alternating methods utilized analytical solutions for both a non-crack plate, as well as for the cracks in an infinite plate. Early applications of analytical alternating method involve the edge crack problem in a semi-infinite plate [13] and the surface crack in a 3D body [14]. The method was used later by O'Donoghue et al. [15] to solve the multiple embedded elliptical cracks in an infinite body. Zhang and Hasebe [16] studied the interactions between rectilinear and circumferential crack in an infinite domain. However, the previous research about analytical alternating method were restricted to the problem of simple geometry. The finite element alternating method (FEAM) and boundary element alternating method (BEAM) extended the applications of alternating technique to complicated cracked geometries. In the absence of crack, FEM or BEM can easily obtain the numerical solution of the non-crack finite body with a coarser mesh.

In this study, we consider the transient thermal conduction problem of a finite plate with multiple arbitrarily located cracks subjected to distributed temperature or thermal flow on the external boundaries. The transient full field temperature distribution of the cracked plate is computed by analytical alternating method. To our knowledge, this is the first application of the alternating technique to the transient thermal conduction problem. On the purpose of comparing with previous results, the steady-state problem of a square plate with two inclined insulated cracks is considered and good agreement is obtained. Finally, results of several transient thermal conduction problems with different boundary conditions are shown and discussed in detail.

2. Analytical fundamental solutions in steady state

In the literature of Schwartz–Neumann alternating method, various types of analytical solutions were used to analyze different problems. Polynomial and Chebyshev polynomial distributions are mostly demonstrated to simulate the continuously distributed loads. For the discontinuously distributed loads, the point loads or piecewise distributed loads could precisely describe the large variation of the crack surface loads in the ultimate case of two closed cracks. For simplicity and versatility, all of the analytical solutions presented in this section are point load solution.

2.1. A finite crack in an infinite plate

Fig. 1 shows an infinite plate with a finite crack lying along the x-axis. The center of the crack is located at the origin and the crack length is $2a$. The crack face is subjected to a point temperature gradient impulse in the y-direction at $x = b$, that is

$$\frac{\partial T}{\partial y} = \delta(x - b), \tag{1}$$

where $T(x, y)$ is the temperature and $\delta(x - b)$ is a unit impulse at $x = b$. The governing equation of a heat conduction problem is similar to the two-dimensional anti-plane mechanics problem. Hence, from the well-known complex variable method, the complex functions are found as

$$Z_{\mathrm{III}} = \frac{1}{\pi} \frac{\sqrt{a^2 - b^2}}{(z - b)\sqrt{z^2 - a^2}}, \tag{2a}$$

$$\bar{Z}_{\mathrm{III}} = \frac{1}{\pi} \sin^{-1}\left[\frac{bz - a^2}{a(z - b)}\right], \tag{2b}$$

where $z = x + \mathrm{i}y$, the full field temperature distribution and temperature gradients for the infinite plate can be expressed as

$$T = \mathrm{Im}\bar{Z}_{\mathrm{III}} = \mathrm{Im}\left\{\frac{1}{\pi} \sin^{-1}\left[\frac{bz - a^2}{a(z - b)}\right]\right\}, \tag{3}$$

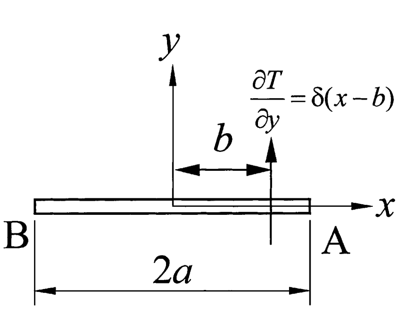


Fig. 1. Point temperature gradient is applied on the crack face of an infinite plate.

$$f_x = \frac{\partial T}{\partial x} = \mathrm{Im}Z_{\mathrm{III}} = \frac{1}{\pi} \frac{\sqrt{a^2 - b^2}}{r_3\sqrt{r_1 r_2}} \sin\left(-\frac{\phi_1 + \phi_2}{2} - \phi_3\right), \tag{4a}$$

$$f_y = \frac{\partial T}{\partial y} = \mathrm{Re}Z_{\mathrm{III}} = \frac{1}{\pi} \frac{\sqrt{a^2 - b^2}}{r_3\sqrt{r_1 r_2}} \cos\left(-\frac{\phi_1 + \phi_2}{2} - \phi_3\right) \tag{4b}$$

in which

$$r_1 = |z - a|, \quad \phi_1 = \tan^{-1}\left(\frac{y}{x - a}\right),$$

$$r_2 = |z + a|, \quad \phi_2 = \tan^{-1}\left(\frac{y}{x + a}\right),$$

$$r_3 = |z - b|, \quad \phi_3 = \tan^{-1}\left(\frac{y}{x - b}\right).$$

For a homogeneous isotropic solid, the heat flux vector is given as $\mathbf{q}(x, y) = -\kappa f_x \mathbf{i} - \kappa f_y \mathbf{j}$, in which κ is the thermal conductivity of the material.The solutions presented in Eqs. (2a), (2b), (3), (4a), (4b) are valid only for an antisymmetric temperature distribution.

For the crack face, that is, subjected to an arbitrarily distributed temperature gradient $f_y(\xi)$, the solution can be constructed by integrating the product of $f_y(\xi)$ and the Green's function of temperature gradient impulse applied on the crack face. Piecewise Gauss's integration is used to precisely describe the high variation of crack face condition. The crack face is divided into 5 sections and 24 Gauss points are distributed on each section. The highly concentrated integral points ensure the precision of the complicated problem particularly for the case while the crack tip is either close to the boundary or another crack tip.

2.2. Rectangular plate with different boundary conditions

Consider a rectangular plate as shown in Fig. 2, w and h are half of the width and height of the plate, respectively. The temperature is zero at the left boundary, and the normal temperature gradient is zero at both the top and bottom boundaries. A temperature impulse is applied on the right boundary, the steady-state heat conduction equation and the boundary conditions for this problem are expressed as follows:

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0, \quad 0 < x < 2w, \ 0 < y < 2h, \tag{5a}$$

$$f_y(x, 0) = \frac{\partial T(x, 0)}{\partial y} = 0, \quad 0 \leqslant x \leqslant 2w, \tag{5b}$$

$$f_y(x, 2h) = \frac{\partial T(x, 2h)}{\partial y} = 0, \quad 0 \leqslant x \leqslant 2w, \tag{5c}$$

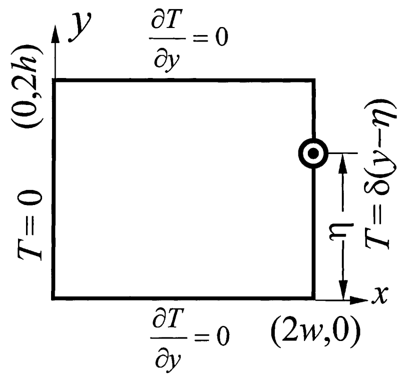


Fig. 2. Point temperature applied on the right boundary of a rectangular plate.

$$T(0,y) = 0, \quad 0 \leqslant y \leqslant 2h, \tag{5d}$$

$$T(2w,y) = \delta(y-\eta), \quad 0 \leqslant y \leqslant 2h. \tag{5e}$$

The solution of full field temperature distribution on the rectangular plate is

$$T(x,y) = \frac{x}{4wh} + \sum_{n=1}^{\infty} \frac{\cos((n\pi\eta)/2h)}{h\sinh((n\pi w)/h)} \times \cos\left(\frac{n\pi y}{2h}\right)\sinh\left(\frac{n\pi x}{2h}\right) \tag{6}$$

and the temperature gradients are

$$f_x(x,y) = \frac{1}{4wh} + \sum_{n=1}^{\infty} \frac{n\pi}{2h^2}\frac{\cos((n\pi\eta)/2h)}{\sinh((n\pi w)/h)} \times \cos\left(\frac{n\pi y}{2h}\right)\cosh\left(\frac{n\pi x}{2h}\right), \tag{7a}$$

$$f_y(x,y) = \sum_{n=1}^{\infty} \frac{-n\pi}{2h^2}\frac{\cos((n\pi\eta)/2h)}{\sinh((n\pi w)/h)} \times \sin\left(\frac{n\pi y}{2h}\right)\sinh\left(\frac{n\pi x}{2h}\right). \tag{7b}$$

Table 1 shows the fundamental solutions of temperature distribution for four types of boundary-value problems. If the temperature impulse or the point temperature gradient is applied on the other boundary, the solution can be obtained from Table 1 by appropriate replacement of variables.

The temperature gradients of the cases I–III in Table 1 can be derived from the presented temperature solutions. For the general case with distributed temperatures $T_r(2w,y)$ and $T_l(0,y)$ on the right and left boundaries, and temperature gradients $f_t(x,2h)$ and $f_b(x,0)$ on the top and bottom boundaries, the full field temperature distribution can be constructed by integrating the solutions shown in Table 1(Ia) and (Ib) as

$$\begin{aligned} S(x,y) = {} & \int_0^{2h} [T_r(2w,\eta)S_{Ia}(x,y,\eta) \\ & + T_l(0,\eta)S_{Ib}(x,y,\eta)]\,\mathrm{d}\eta \\ & + \int_0^{2w} [f_t(\xi,2h)S_{Ic}(x,y,\xi) \\ & + f_b(\xi,0)S_{Id}(x,y,\xi)]\,\mathrm{d}\xi, \end{aligned} \tag{8}$$

where $S_{Ia}(x,y,\eta), S_{Ib}(x,y,\eta), S_{Ic}(x,y,\xi)$, and $S_{Id}(x,y,\xi)$ are the fundamental solutions presented in Table 1 (Ia) and (Ib). A piecewise Gauss's integration is utilized in Eq. (8), each side of the plate is divided into 5 sections and Gauss points are distributed on each section.

3. Analytical fundamental solutions of transient problems for rectangular plates

If the transient heat conduction problem is non-homogeneous due to the non-homogeneity of boundary conditions, the original problem can be decomposed into several simple problems that can be solved by the separation of variables [17]. They are:

(i) A non-homogeneous steady-state problem with solution described by $T_s(x,y)$.

(ii) A homogeneous transient problem with solution described by $T_h(x,y,t)$.

A rectangular plate as shown in Fig. 3 is considered in the transient analysis, where $2w$ and $2h$ are the width and height of the plate, respectively. The mathematical formulation of this problem is given as

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t}, \quad 0 < x < 2w,\ 0 < y < 2h, \tag{9a}$$

$$\frac{\partial T}{\partial x} = f_0, \quad x = 0,\ 0 \leqslant y \leqslant 2h, \tag{9b}$$

$$\frac{\partial T}{\partial x} = f_0, \quad x = 2w,\ 0 \leqslant y \leqslant 2h, \tag{9c}$$

$$T = -T_0, \quad y = 0,\ 0 \leqslant x \leqslant 2w, \tag{9d}$$

$$T = T_0, \quad y = 2h,\ 0 \leqslant x \leqslant 2w, \tag{9e}$$

$$T = F(x,y), \quad \text{for } t = 0,\ 0 < x < 2w,\ 0 < y < 2h, \tag{9f}$$

where α is the thermal diffusivity.

The original problem can be decomposed into a non-homogeneous steady-state problem given by

$$\frac{\partial^2 T_s}{\partial x^2} + \frac{\partial^2 T_s}{\partial y^2} = 0, \quad 0 < x < 2w,\ 0 < y < 2h \tag{10a}$$

$$\frac{\partial T_s}{\partial x} = f_0, \quad x = 0,\ 0 \leqslant y \leqslant 2h, \tag{10b}$$

$$\frac{\partial T_s}{\partial x} = f_0, \quad x = 2w,\ 0 \leqslant y \leqslant 2h, \tag{10c}$$

$$T_s = -T_0, \quad y = 0,\ 0 \leqslant x \leqslant 2w \tag{10d}$$

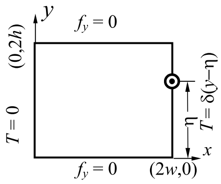
$$T_s = T_0, \quad y = 2h,\ 0 \leqslant x \leqslant 2w \tag{10e}$$

and a homogeneous transient problem given as

Table 1
Solutions for four types of boundary-value problem for rectangular plates

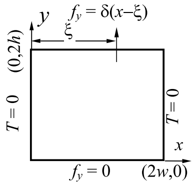
Neumann condition on top and bottom boundaries, Dirichlet condition on right and left boundaries

Ia.



$$T(x,y)=\frac{x}{4wh}+\sum_{n=1}^{\infty}\frac{\cos((n\pi\eta)/(2h))}{h\sinh((n\pi w)/h)}\cos\left(\frac{n\pi y}{2h}\right)\sinh\left(\frac{n\pi x}{2h}\right)$$

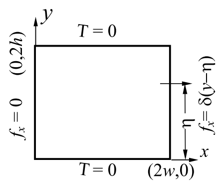
Ib.



$$T(x,y)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\frac{\sin((n\pi\xi)/(2w))}{\sinh((n\pi h)/w)}\sin\left(\frac{n\pi x}{2w}\right)\cosh\left(\frac{n\pi y}{2w}\right)$$

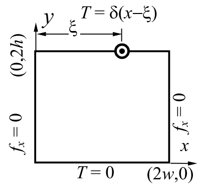
Dirichlet condition on top and bottom boundaries, Neumann condition on right and left boundaries

IIa.



$$T(x,y)=\sum_{n=1}^{\infty}\frac{2}{n\pi}\frac{\sin((n\pi\eta)/(2h))}{\sinh((n\pi w)/h)}\sin\left(\frac{n\pi y}{2h}\right)\cosh\left(\frac{n\pi x}{2h}\right)$$

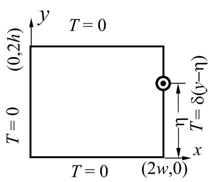
IIb.



$$T(x,y)=\frac{y}{4wh}+\sum_{n=1}^{\infty}\frac{\cos((n\pi\xi)/(2w))}{w\sinh((n\pi h)/w)}\cos\left(\frac{n\pi x}{2w}\right)\sinh\left(\frac{n\pi y}{2w}\right)$$

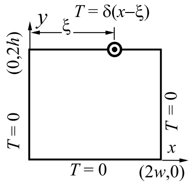
Dirichlet condition on four boundaries

IIIa.



$$T(x,y)=\sum_{n=1}^{\infty}\frac{\sin((n\pi\eta)/(2h))}{h\sinh((n\pi w)/h)}\sin\left(\frac{n\pi y}{2h}\right)\sinh\left(\frac{n\pi x}{2h}\right)$$

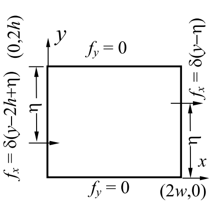
IIIb.



$$T(x,y)=\sum_{n=1}^{\infty}\frac{\sin((n\pi\xi)/(2w))}{w\sinh((n\pi h)/w)}\sin\left(\frac{n\pi x}{2w}\right)\sinh\left(\frac{n\pi y}{2w}\right)$$

Neumann condition on four boundaries

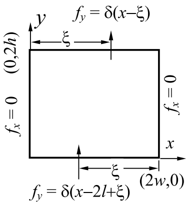
IVa.



$$f_x(x,y)=\frac{1}{2h}+\sum_{n=1}^{\infty}\frac{\cos((n\pi\eta)/(2h))}{h\sinh((n\pi w)/h)}\cos\left(\frac{n\pi y}{2h}\right)\times\left[\sinh\left(\frac{n\pi x}{2h}\right)-(-1)^n\sinh\left(\frac{n\pi(x-2w)}{2h}\right)\right]$$

(*continued on next page*)

Table 1 (*continued*)

Neumann condition on top and bottom boundaries, Dirichlet condition on right and left boundaries		
		$f_y(x,y) = \sum_{n=1}^{\infty} \frac{-\cos((n\pi\eta)/(2h))}{h\sinh((n\pi w)/h)} \sin\left(\frac{n\pi y}{2h}\right) \times \left[\cosh\left(\frac{n\pi x}{2h}\right) - (-1)^n \cosh\left(\frac{n\pi(x-2w)}{2h}\right)\right]$
IVb.		$f_x(x,y) = \sum_{n=1}^{\infty} \frac{-\cos((n\pi\xi)/(2w))}{w\sinh((n\pi h)/w)} \sin\left(\frac{n\pi x}{2w}\right) \times \left[\cosh\left(\frac{n\pi y}{2w}\right) - (-1)^n \cosh\left(\frac{n\pi(y-2h)}{2w}\right)\right]$ $f_y(x,y) = \frac{1}{2w} + \sum_{n=1}^{\infty} \frac{\cos((n\pi\xi)/(2w))}{w\sinh((n\pi h)/w)} \cos\left(\frac{n\pi x}{2w}\right) \times \left[\sinh\left(\frac{n\pi y}{2w}\right) - (-1)^n \sinh\left(\frac{n\pi(y-2h)}{2w}\right)\right]$

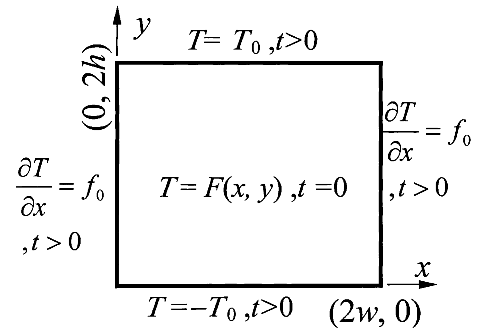


Fig. 3. The transient problem of a rectangular plate with prescribed boundary and initial conditions.

$$\frac{\partial^2 T_h}{\partial x^2} + \frac{\partial^2 T_h}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T_h}{\partial t}, \quad 0 < x < 2w, \ 0 < y < 2h, \tag{11a}$$

$$\frac{\partial T_h}{\partial x} = 0, \quad x = 0, \ 0 \leqslant y \leqslant 2h, \tag{11b}$$

$$\frac{\partial T_h}{\partial x} = 0, \quad x = 2w, \ 0 \leqslant y \leqslant 2h, \tag{11c}$$

$$T_h = 0, \quad y = 0, \ 0 \leqslant x \leqslant 2w, \tag{11d}$$

$$T_h = 0, \quad y = 2h, \ 0 \leqslant x \leqslant 2w \tag{11e}$$

with initial condition $T_h = F(x,y) - T_s(x,y) \equiv F^*(x,y)$, for $t = 0, \ 0 < x < 2w, \ 0 < y < 2h$. The solution for the original problem of Eqs. (9a)–(9f) is determined from the two solutions, $T_s(x,y)$ and $T_h(x,y,t)$, as

$$T(x,y,t) = T_s(x,y) + T_h(x,y,t) \tag{12}$$

The solution of the steady-state problem described in Eqs. (10a)–(10e) is given as

$$T_s(x,y) = -T_0 + \frac{T_0}{h}y + \sum_{k=1}^{\infty} \frac{4hf_0}{(k\pi)^2}[1-(-1)^k] \times \frac{\sin((k\pi y)/2h)}{\sinh((k\pi w/h)} \left[\cosh\left(\frac{k\pi x}{2h}\right) - \cosh\left(\frac{k\pi(x-2w)}{2h}\right)\right] \tag{13}$$

The general solution of the transient problem described in Eqs. (11a)–(11e) can be written as

$$T_h(x,y,t) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn}\, e^{-\alpha(\beta_m^2+\gamma_n^2)t} X(\beta_m,x) Y(\gamma_n,y), \tag{14}$$

where the eigenfunctions $X(\beta_m,x)$ and $Y(\gamma_n,y)$ and eigenvalues β_m and γ_n are given by

$$X(\beta_m,x) = \cos(\beta_m x), \ \ \beta_m = \frac{m\pi}{2w}, \ \ m = 0,1,2,\ldots,\infty, \tag{15a}$$

$$Y(\gamma_n,x) = \cos(\gamma_n x), \quad \gamma_n = \frac{n\pi}{2h}, \quad n = 1,2,\ldots,\infty. \tag{15b}$$

The initial condition is rewritten as

$$T_h = F(x,y) - T_s(x,y) = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty} c_{mn} X(\beta_m,x) Y(\gamma_n,y). \tag{16}$$

The unknown coefficient c_{mn} can be determined by operating both sides of Eq. (16) using operators

$$\int_0^{2w} X(\beta_m,x)\,dx, \quad \int_0^{2h} Y(\gamma_n,y)\,dy. \tag{17}$$

By utilizing the orthogonality of eigenfunctions, c_{mn} is expressed as

$$c_{mn} = \frac{1}{M(\beta_m)N(\gamma_n)} \int_0^{2h} \int_0^{2w} \left[F(x',y') - T_s(x',y')\right] \times \cos(\beta_m x') \sin(\gamma_n y')\, dx'\, dy', \tag{18}$$

where

$$M(\beta_m) = \int_0^{2w} \cos^2\left(\frac{m\pi x'}{2w}\right) dx' = \begin{cases} 2w, & m = 0 \\ w, & m \neq 0 \end{cases}$$

$$N(\gamma_n) = \int_0^{2h} \sin^2\left(\frac{n\pi y'}{2h}\right) dy' = h, \quad n \neq 0$$

Special case. If the initial temperature is zero (i.e. $F(x,y) = 0$) in the whole region, the coefficient becomes

$$c_{0n} = \frac{1}{2wh} \int_0^{2h} \int_0^{2w} -T_s(x,y) \sin\left(\frac{n\pi y'}{2h}\right) dx'\, dy' = \frac{2T_0}{n\pi}[1 + (-1)^n], \quad n = 1, 2, \ldots, \infty, \tag{19a}$$

$$c_{mn} = \frac{1}{wh} \int_0^{2h} \int_0^{2w} -T_s(x,y) \cos\left(\frac{m\pi x'}{2w}\right) \times \sin\left(\frac{n\pi y'}{2h}\right) dx'\, dy' = \sum_{k=1}^{\infty} [1 - (-1)^k][1 - (-1)^m] \frac{8f_0}{k\pi^3} \times \frac{wh^2 \delta_{nk}}{(h^2m^2 + w^2k^2)}, \tag{19b}$$

$$m = 1, 2, \ldots, \infty, \quad n = 1, 2, \ldots, \infty$$

and the homogeneous transient solution is

$$T_h(x,y,t) = \sum_{n=1}^{\infty} \frac{2T_0}{n\pi}[1 + (-1)^n] \sin\left(\frac{n\pi y}{2h}\right) e^{-\alpha\{(n\pi)/(2h)\}^2 t} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [1 - (-1)^m][1 - (-1)^n] \frac{8f_0}{n\pi^3} \times \frac{wh^2}{(h^2m^2 + w^2n^2)} \cos\left(\frac{m\pi x}{2w}\right) \times \sin\left(\frac{n\pi y}{2h}\right) e^{-\alpha[\{(m\pi)/(2w)\}^2 + \{(n\pi)/(2h)\}^2]t}. \tag{20}$$

The original problem with zero initial temperature becomes

$$T(x,y,t) = -T_0 + \frac{T_0}{h} y + \sum_{k=1}^{\infty} \frac{4hf_0}{(k\pi)^2} \times [1 - (-1)^k] \frac{\sin((k\pi y)/(2h))}{\sinh((k\pi w)/h)} \times \left[\cosh\left(\frac{k\pi x}{2h}\right) - \cosh\left(\frac{k\pi(x - 2w)}{2h}\right)\right] + \sum_{n=1}^{\infty} \frac{2T_0}{n\pi}[1 + (-1)^n] \sin\left(\frac{n\pi y}{2h}\right) e^{-\alpha\{(n\pi)/(2h)\}^2 t} + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} [1 - (-1)^m][1 - (-1)^n] \frac{8f_0}{n\pi^3} \times \frac{wh^2}{(h^2m^2 + w^2n^2)} \cos\left(\frac{m\pi x}{2w}\right) \times \sin\left(\frac{n\pi y}{2h}\right) e^{-\alpha[\{(m\pi)/(2w)\}^2 + \{(n\pi)/(2h)\}^2]t} \tag{21}$$

4. Analytical alternating method for transient heat conduction problems with multiple cracks

In this section, the analytical alternating method will be developed to analyze the transient thermal conduction problem with multiple cracks. In order to illustrate the proposed alternating procedure, a rectangular plate with multiple insulated cracks in Fig. 4(a) is considered. At time $t = 0$, constant temperatures T_t and T_b are applied on the top and bottom boundaries, and normal temperature gradients f_r and f_l are applied on the right and left boundaries.The initial temperature is zero in the full field of the cracked plate. By the principle of superposition, this complicated transient heat conduction problem can be solved by the iteration of several simple problems as indicated below.

1. At time t, the transient temperature distribution of the original problem (Fig. 4(a)) can be superposed by a transient problem without crack and a steady-state problem with cracks as shown in Figs. 4(b)

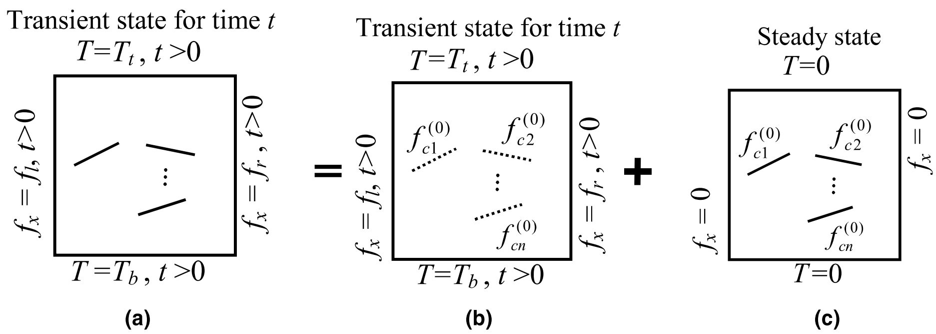


Fig. 4. The alternating method of transient heat conduction.

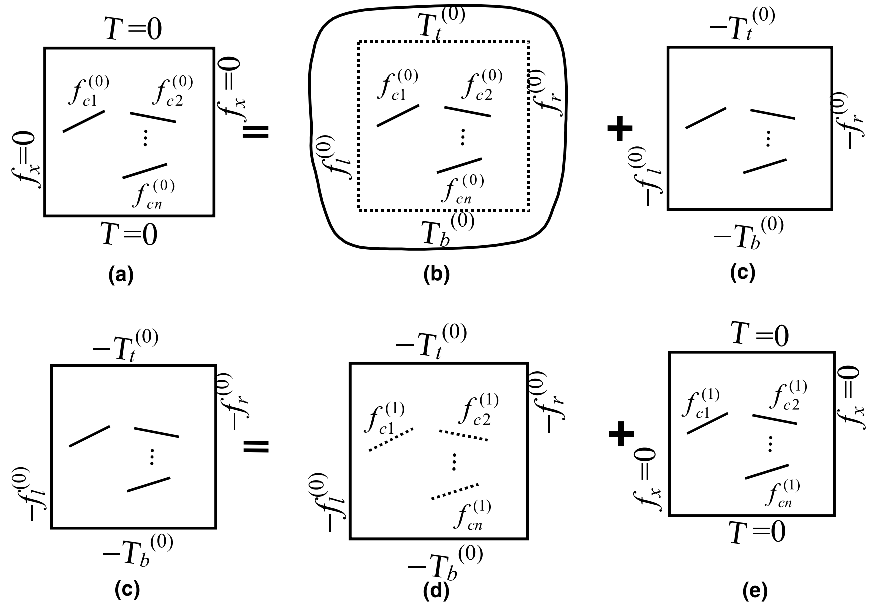


Fig. 5. Alternating method of a finite plate with multiple cracks.

and (c), respectively. First, solve the transient non-crack plate in Fig. 4(b) with the same geometry and boundary conditions as the original problem of Fig. 4(a), this problem has been discussed in the previous section. Evaluate the residual normal temperature gradient $f_{ci}^{(0)}(i = 1, 2, \ldots, n)$ at the locations of the fictitious crack i for time t. Since the crack's faces are insulated, the normal temperature gradient $f_{ci}^{(0)}$ at cracks $i = 1, 2, \ldots, n$ are superposed as shown in Fig. 4(c) to satisfy the zero heat flux across the crack.

2. The steady-state problem of the cracked plate as shown in Fig. 4(c) (or Fig. 5(a)) can be further split into the following problems.

   (a) Fig. 5(b) shows an infinite plate with multiple cracks and subjected to the same temperature gradients on the crack faces as shown in Fig. 5(a). Evaluate the residual external boundary conditions $(T_t^{(0)}, T_b^{(0)}, f_r^{(0)}, f_l^{(0)})$ on the virtual boundaries as the original plate by the internal iteration procedure of the alternating method which will be described later. The residual boundary conditions could be superposed reversely by Fig. 5(c).

   (b) Now, the solution of Fig. 5(c) can be obtained by the solutions of Figs. 5(d) and (e). Fig. 5(d) is the steady-state problem of the non-crack plate with distributed external boundary conditions $(-T_t^{(0)}, -T_b^{(0)}, -f_r^{(0)}, -f_l^{(0)})$. Utilizing the analytical solutions presented in Table 1(IIa) and (IIb), the solution of Fig. 5(d) can be obtained by integrating the product of the Green's function solutions and the distributed external boundary conditions $(-T_t^{(0)}, -T_b^{(0)}, -f_r^{(0)}, -f_l^{(0)})$. The problem for applied residual temperature gradient $f_{ci}^{(1)}$ on the crack face in Fig. 5(e) will be preceeded in the next (external) iteration of alternating procedure.

3. In the iteration procedure, the solution of an infinite plate with multiple cracks as shown in Fig. 5(b) can be constructed by the procedure indicated in Fig. 6, in which $f_{ci}^{(0)}$ in Fig. 6(a) is the normal temperature gradient of the crack i of an infinite plate. The solution of Fig. 6(a) could be obtained by the following internal iterations.

   (a) Consider an infinite plate containing a single crack j and the crack face is subjected to a normal temperature gradient $f_{cj}^{(0)}$, and evaluates the normal temperature gradient $f_{cij}^{(0)}$ of the fictitious crack i.

   (b) Superpose the total solutions of the problems with single crack i ($i = 1$–n) and the residual temperature gradient of crack i is found as

$$g_{ci}^{(1)} = \sum_{\substack{j=1\\ j\neq i}}^{n} f_{cij}^{(0)} \tag{22}$$

   for the k'th cycle of the internal iteration

$$g_{ci}^{(k+1)} = \sum_{\substack{j=1\\ j\neq i}}^{n} g_{cij}^{(k)} \tag{23}$$

   (c) After several cycle of iterations, the residual temperature gradient $g_{ci}^{(k+1)}$ approaches zero, then the solution of Fig. 6(a) could be obtained by

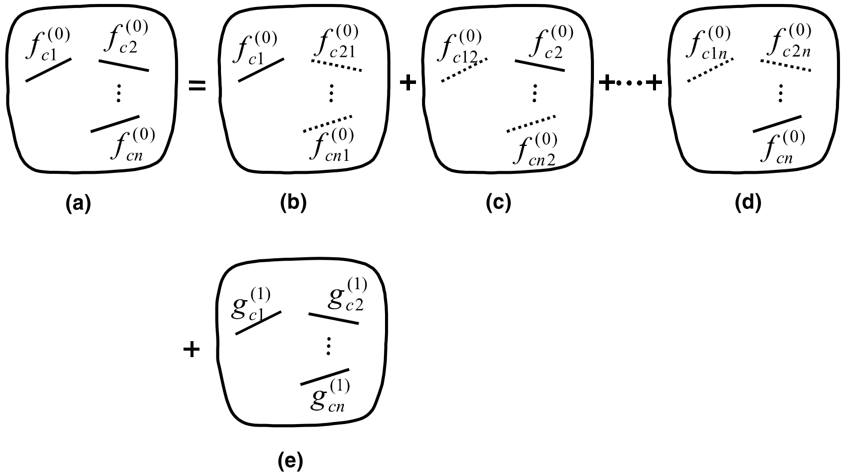


Fig. 6. Alternating method of an infinite plate with multiple cracks.

combining the solutions for infinite plates with a single crack subjected to the normal temperature gradient g_{ci} as

$$g_{ci} = f_{ci}^{(0)} + \sum_{k=1}^{\infty} g_{ci}^{(k)} \tag{24}$$

(d) The convergent criterion is taken as

$$\frac{g_{ci}^{(k+1)} - g_{ci}^{(k)}}{g_{ci}^{(k)}} < 10^{-5}, \quad i = 1\text{–}n. \tag{25}$$

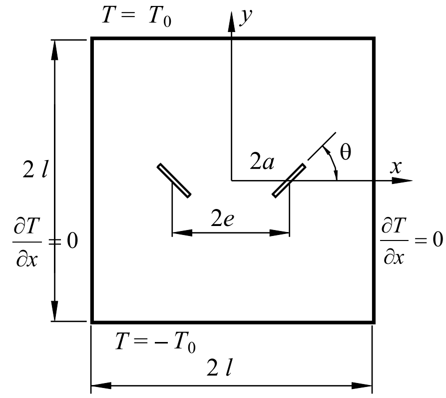


Fig. 7. A square plate with two inclined insulated cracks.

5. Numerical results and discussions

Example 1 (*Steady-state temperature distribution of a finite plate with two inclined insulated cracks*). A square plate containing two inclined insulated cracks that are located symmetrically to the y-axis and subjected to constant temperatures T_0 and $-T_0$ on the top and bottom boundaries is considered first. The right and left boundaries are insulated. The length of these two cracks is $2a$ and the width of the square plate is $2l$. The distance between the centers for these two cracks is $2e$ and θ is the inclined angle of the crack. The geometric configuration of this problem is shown in Fig. 7. The temperature fields for the cases of $a/l = 0.3, e/l = 0.33$, and $\theta = 0°, 60°, 90°, 120°$ are investigated, and the contours of constant temperature are shown in Fig. 8. Chen and Chang [10] also analyzed the same problem by finite element alternating method in which 24 eight-node isoparametric quadrilateral elements were used to compute the non-cracked finite plate, and a third-order polynomial function was used to simulate the crack-face heat flux in an infinite plate. For the present calculations, 120 Gauss's integral points are distributed on each side of external boundaries and on each crack face. Excellent agreements are found between the present analytical alternating method and the finite element alternating method, as shown in Fig. 8.

Example 2 (*Steady-state temperature distribution of a finite plate with two insulated cracks and prescribed temperature on four sides of the plate*). In order to examine the interaction between cracks and the boundaries of a plate, a square plate with two insulated cracks is subjected to $T = T_0$ and $T = -T_0$ at the top and bottom boundaries and $T = 0$ on both the right and left boundaries is considered. The geometric configuration of this unsymmetrical problem is shown in Fig. 9. The

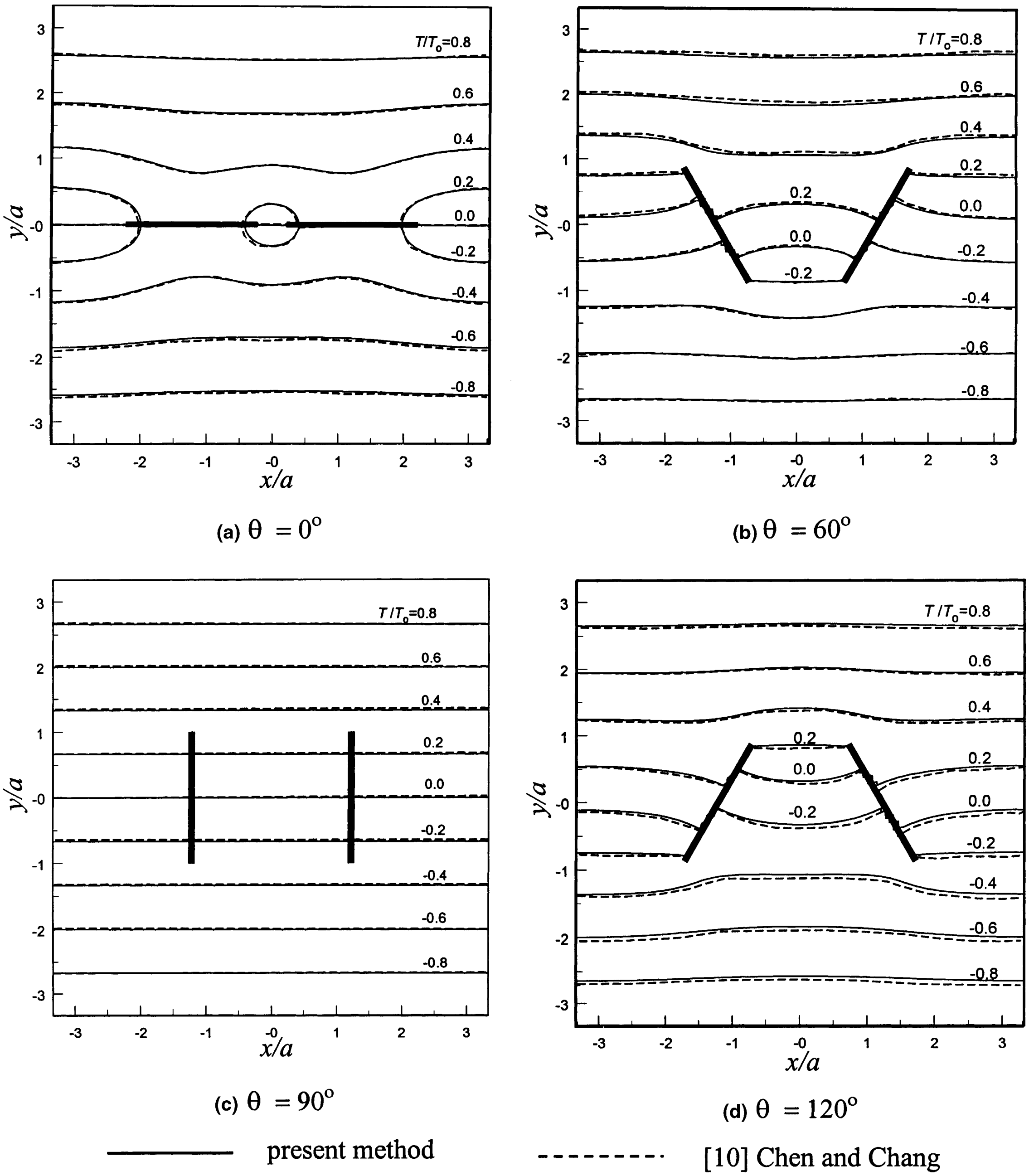


Fig. 8. The steady-state temperature distribution of a plate with two inclined insulated cracks.

length of the square plate is $2l$ and lengths of both cracks are 2 a, the centers of the two cracks are located at $(-a/2, -a/2)$ and $(a/2, a/2)$, respectively. The full field temperature distributions for the cases of $l/a = 2, \theta^{\circ} = 0^{\circ}$ and 45° are shown in Fig. 10. Since the crack length is long, the temperature distributions are strongly influenced by the interactions between two cracks and boundaries. The high intensity of the temperature contours around the corners of the plate in Fig. 10 interprets the discontinuity of the temperature around the corners under the given temperature conditions.

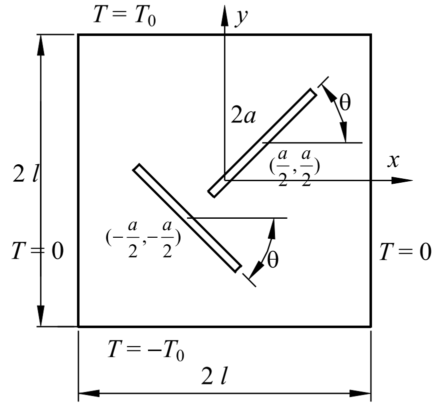


Fig. 9. The unsymmetrical steady-state problem of a square plate with two inclined insulated cracks.

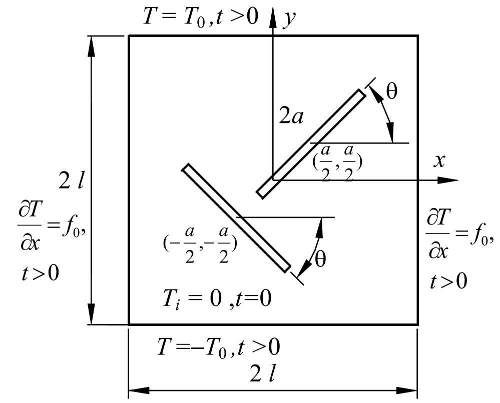


Fig. 11. Transient analysis of a square plate with two inclined insulated cracks ($f_0 = T_0/l$).

Example 3 (*Transient temperature distribution of a finite plate with two inclined insulated cracks*). The cases of a finite plate with two arbitrarily located insulated cracks are presented to demonstrate the versatility of this method. A square plate with two inclined insulated cracks as shown in Fig. 11 is considered. The initial temperature of the plate is $T_i = 0$. At time $t = 0$, the top and bottom boundaries are subjected to constant temperatures of $T = T_0$ and $T = -T_0$, respectively, and both the right and left boundaries are subjected to the temperature gradient of $\partial T/\partial x = f_0$ The solutions for the case of $l/a = 2, \theta = \pi/4$ and $f_0 = T_0/l$ (the crack centers are respectively located at $(-a/2, -a/2)$ and $(a/2, a/2)$ are calculated. As shown in Figs. 12–14, the transient temperature and transient temperature gradient approach the steady-state solution at dimensionless time $(t\alpha)/a^2 = 4$. The high intensity of temperature gradient contours around the crack tips in Figs. 13 and 14

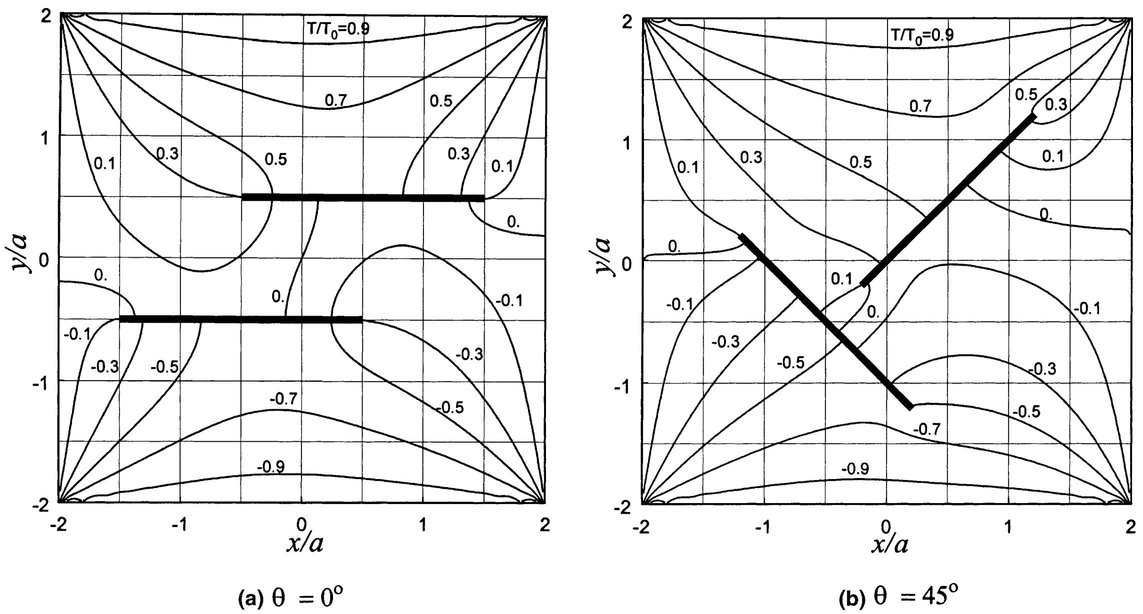


Fig. 10. The steady-state temperature distribution of a plate with two inclined insulated cracks for prescribed temperature on four boundaries of the plate.

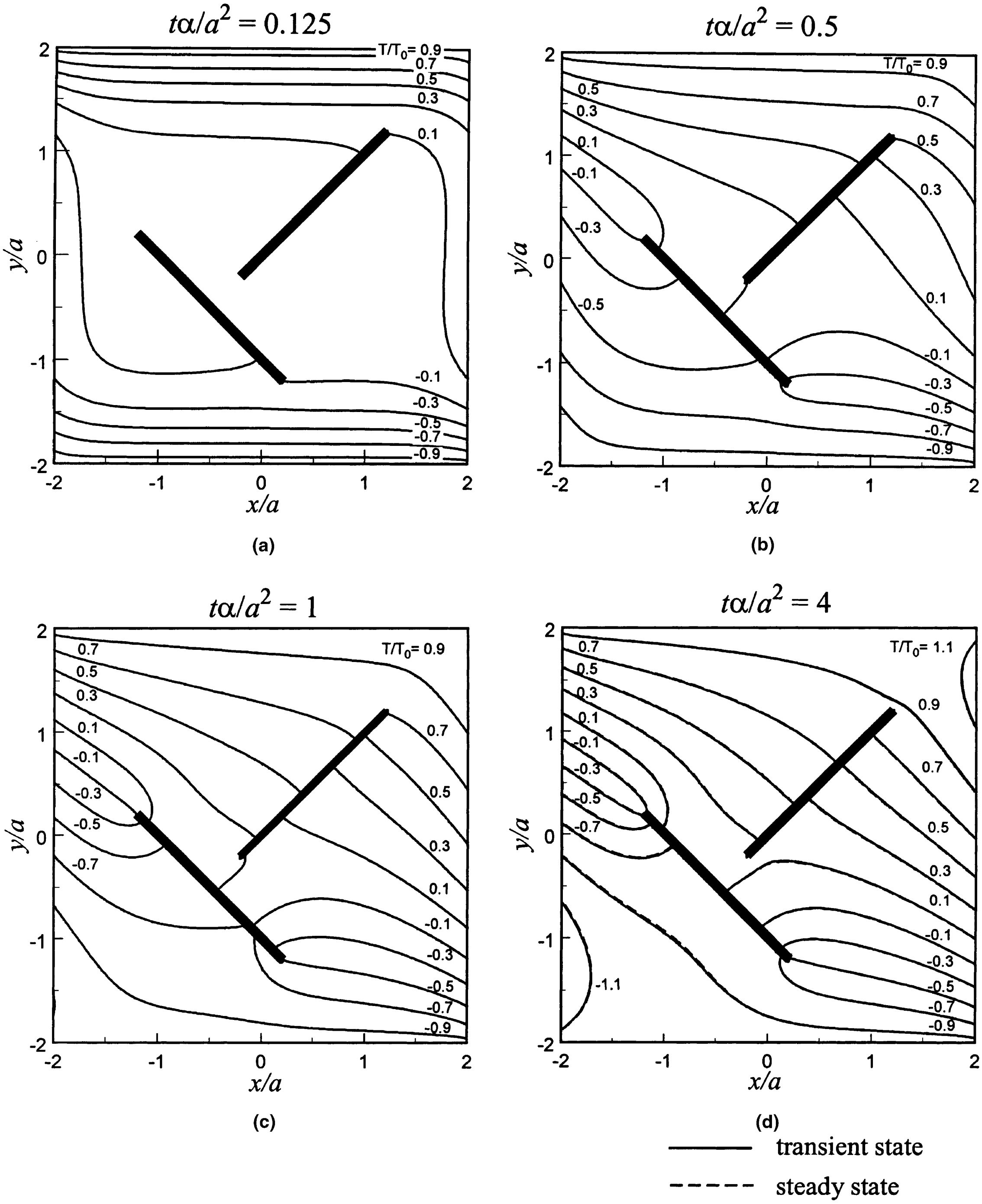


Fig. 12. The transient temperature of a plate with two inclined insulated cracks ($f_0 = T_0/l$).

interpret the singularity of the temperature gradient around the crack tips. In order to provide a high resolution temperature contour, a total number 14 400 (120 × 120) points on the cracked plate are computed. For this complicated case, the computing time is less then 20 min on an Intel Pentium II 300 MHz PC.

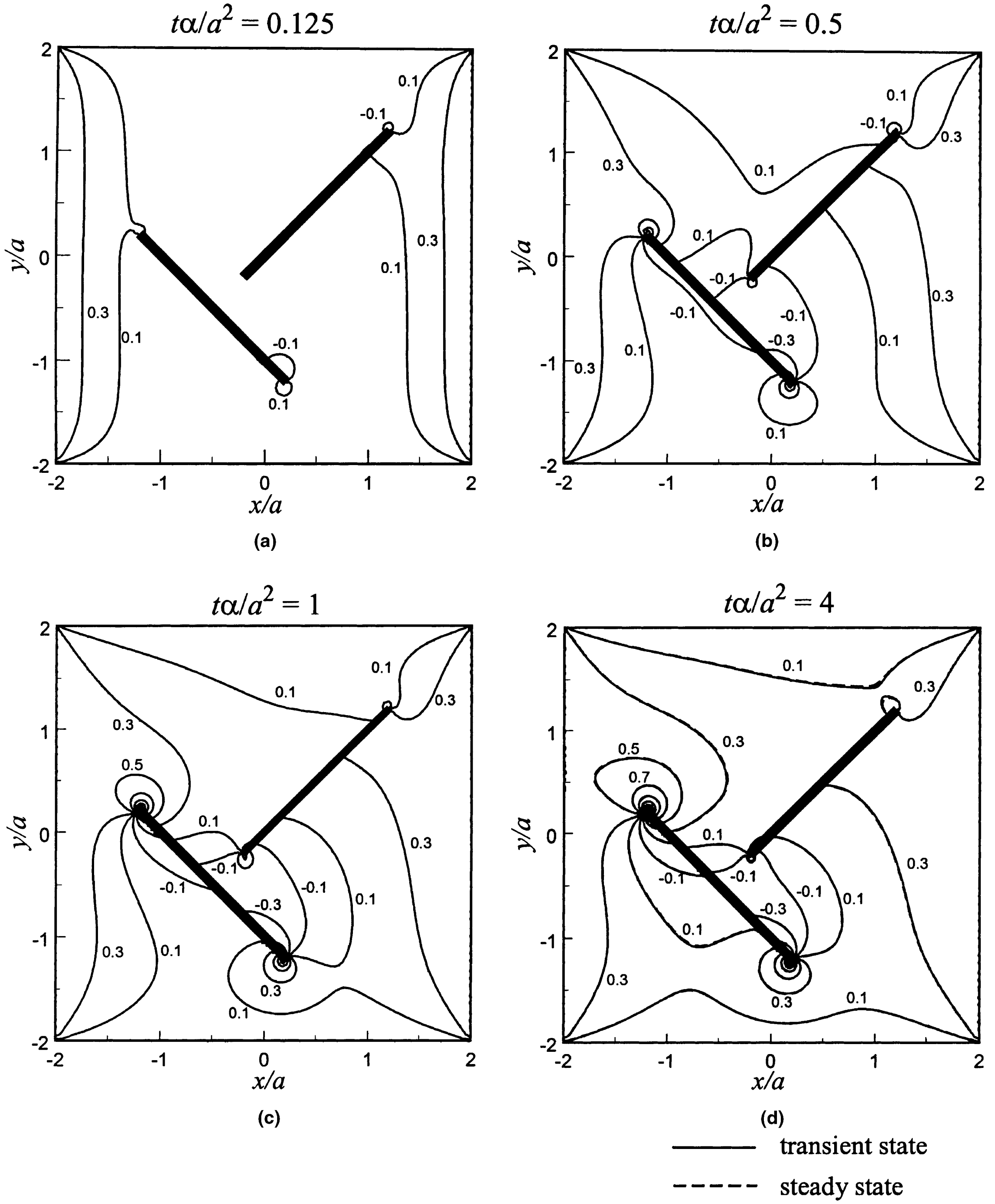


Fig. 13. The transient temperature gradient (x-direction) of a plate with two inclined insulated cracks ($f_0 = T_0/l$).

6. Conclusion

By using the predetermined analytical fundamental solutions, an analytical alternating method is utilized to investigate the transient conduction problem of a finite plate with multiple cracks. All the steady-state fundamental solutions used in this study are Green function solutions that can accurately describe the large variation of the temperature gradient on the crack faces and the boundary conditions of the finite plate. An efficient and

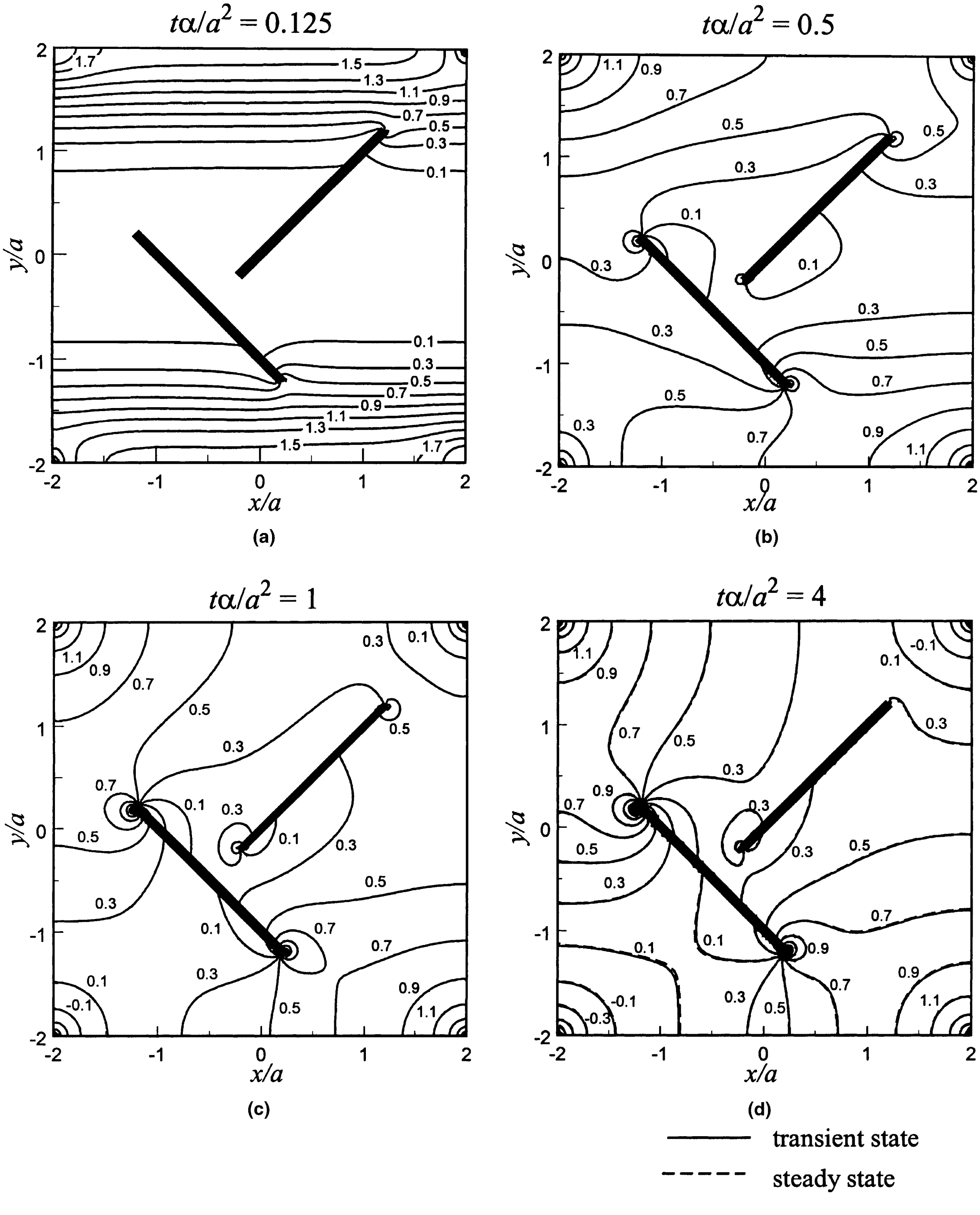


Fig. 14. The transient temperature gradient (y-direction) of a plate with two inclined insulated cracks ($f_0 = T_0/l$).

accurate Gauss's integration method has been successfully developed to analyze the transient problem of a finite plate with arbitrarily located multiple cracks. For the numerical results of steady-state case, excellent consistency between the present results and the available reference solution is achieved. In the transient analysis, the transient temperature distributions are computed for cases of multiple cracks and different boundary conditions. The interaction between the cracks and the boundaries of the finite plate is observed and the characteristic time for the transient solution approaches the steady-state solution is also discussed.

Acknowledgements

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